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Construction of Improved Stress-Energy Tensor in $d \geq 2$

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The weakest set of necessary and sufficient conditions for the existence of a conserved traceless symmetric stress-energy tensor for a scale-invariant Poincaré field theory over a d-dimensional flat space-time manifold has been proved. This improved tensor is explicitly constructed for arbitrary space-time dimensions including $d = 2$. The example of a vector field in $d \geq 3$, $d \neq 4$ demonstrates that full conformal symmetry over a flat space-time is sufficient but not necessary for the existence of an improved tensor.

1. INTRODUCTION

Scale invariance is a constraint on the trace of conserved stress-energy tensors. Coleman and others (Callan *et al.*, 1970; Coleman and Jackiw, 1971) found a set of necessary and sufficient conditions for modifying the conventional stress-energy tensors to a traceless symmetric conserved improved tensor over a (3 + 1)-dimensional flat space-time manifold. The question of the existence of an improved tensor for a 2-dimensional classical field theory has been a long-standing open problem (Francesco *et al.*, 1997). In this paper we will discuss constructive proofs of all the necessary and sufficient conditions for the existence of an improved tensor for a local scale- and Poincaré-invariant Lagrangian field theory in an arbitrary-dimensional space-time including $d = 2$.

2. SPACE-TIME SYMMETRIES

We choose the fields $\varphi^A(x)$ as various representations of the Lorentz group in a d-dimensional flat space-time manifold M. In an infinitesimal

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proper orthochronous Lorentz transformation $x^\mu \to x'^\mu = x^\mu + \omega^\mu_\nu x^\nu$, $|\omega^\mu_\nu| \ll 1$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ the fields transform like

$$\varphi^A(x) \to \varphi'^A(x') = [\delta^A_B + \tfrac{1}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^A_B]\varphi^B(x) \tag{2.1}$$

where $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$ are the spin matrices. $\varphi^A(x)$ transforms like a scalar under space-time translation $x^\mu \to x'^\mu = x^\mu + a^\mu$. Invariance of the action $S[\varphi] = \int_{\Lambda \subseteq M} d^d x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$ under the full Poincaré group defines the following conserved canonical Noether currents:

$$\Theta^{\mu\nu} = \Pi^\mu_A \partial^\nu \varphi^A - \eta^{\mu\nu}\mathcal{L} \tag{2.2}$$

$$M^{\mu\lambda\rho} = (x^\lambda \Theta^{\mu\rho} - x^\rho\, \Theta^{\mu\lambda}) + \Pi^\mu_A (\Sigma^{\lambda\rho})^A_B \varphi^B \tag{2.3}$$

where

$$\Pi^\mu_A = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^A)}, \qquad \eta^{\mu\nu} \equiv \eta_{\mu\nu} = diag\Big(+1, \underbrace{-1, -1, \ldots, -1}_{(d-1)\ \text{times}}\Big)$$

Although $\partial_\mu\Theta^{\mu\nu} \equiv 0$ due to the equation of motion, the conservation of canonical angular momentum $\partial_\mu M^{\mu\lambda\rho} = 0$ is not an identity for arbitrary Lagrangians. Since Noether's conserved currents are not unique, addition of an antisymmetric divergence to $\Theta^{\mu\nu}$ defines a new conserved current which gives the same physical observables, namely the energy and momentum of the field. Belinfante (1939) proved that Lorentz invariance is sufficient for the existence of an antisymmetric tensor field $B^{\alpha\mu\nu} = \frac{1}{2}(\Pi^\alpha\Sigma^{\mu\nu} + \Pi^\mu\Sigma^{\nu\alpha} - \Pi^\nu\Sigma^{\alpha\mu})\varphi$ such that the conserved stress-energy tensor

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu} \tag{2.4}$$

is symmetric (Wentzel, 1949). Conservation of the angular momentum tensor ($J^{\mu\lambda\rho} = x^\lambda T^{\mu\rho} - x^\rho T^{\mu\lambda}$) is now an identity.

Under a scale transformation $x^\mu \to x'^\mu = e^\sigma x^\mu$, $\sigma \in \mathcal{R}$, the fields are expected to transform like

$$\varphi^A(x) \to \varphi'^A(x') = (e^{-\mathbf{D}\sigma})^A_B \varphi^B(x) \tag{2.5}$$

where the dimension matrix $\mathbf{D}$ multiplies every Bose field by $(\frac{d}{2} - 1)$ and every Fermi field by $(d - 1)/2$. The conserved dilatation current

$$d^\mu = x_\nu T^{\mu\nu} + \Pi^\mu \mathbf{D}\varphi + \Pi_\nu \Sigma^{\mu\nu}\varphi \tag{2.6}$$

is due to the invariance of the action over arbitrary $\Lambda \subseteq M$ under scale transformation. The conservation law $\partial_\mu d^\mu = 0$ imposes a constraint on the trace of the stress-energy tensor

$$T^\mu_\mu = -\partial_\mu(\Pi^\mu \mathbf{D}\varphi + \Pi_\nu\Sigma^{\mu\nu}\varphi) \tag{2.7}$$

We wish to use this constraint for modifying $T^{\mu\nu}$ to a traceless tensor.

3. IMPROVED STRESS-ENERGY TENSOR

Any new candidate for stress-energy tensor must (1) be conserved, (2) define the same energy-momentum, and (3) be symmetric. From the inverse Poincaré lemma, $T^{\mu\nu}$ can be improved to some $E^{\mu\nu}$ if and only if there exists a fourth-rank tensor field $C^{\lambda\rho\mu\nu}(x)$ such that

$$\begin{aligned} E^{\mu\nu} &= T^{\mu\nu} + \tfrac{1}{2}\partial_\lambda\partial_\rho C^{\lambda\rho\mu\nu} \\ C^{\lambda\rho\mu\nu} &= -C^{\lambda\mu\rho\nu} \\ C^{\lambda\rho\mu\nu} &= -C^{\nu\rho\mu\lambda} \\ \partial_\lambda\partial_\rho C^{\lambda\rho\mu\nu} &= \partial_\lambda\partial_\rho C^{\lambda\rho\nu\mu} \end{aligned} \tag{3.1}$$

The $C^{\lambda\rho\mu\nu}$ have only one nonzero independent component in $d = 2$. Therefore in a 2-dimensional flat space-time manifold an improved stress-energy tensor exists if and only if there is a scalar field $C^{0101}(x)$ such that

$$E^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{2}\varepsilon^{\mu\lambda}\varepsilon^{\nu\rho}\,\partial_\lambda\partial_\rho C^{0101} \tag{3.2}$$

where $\varepsilon^{\mu\nu}$ is the totally antisymmetric pseudotensor with $\varepsilon^{01} = +1$.

Necessary and Sufficient Condition for $E^{\mu}_{\mu} = 0$ in $d \geq 2$. For a Poincaré- and scale-invariant classical field theory over a flat space-time manifold of $d \geq 2$ a traceless improved tensor exists if and only if there is a tensor field $\sigma^{\mu\nu}(x)$ such that

$$\Pi^{\mu}\mathbf{D}\varphi + \Pi_{\nu}\Sigma^{\mu\nu}\varphi = \partial_\nu\sigma^{\mu\nu} \tag{3.3}$$

where for $d = 2$ there must exist a scalar field $\psi(x)$ such that

$$\sigma_{+}^{\mu\nu} \equiv \tfrac{1}{2}(\sigma^{\mu\nu} + \sigma^{\nu\mu}) = \tfrac{1}{2}\eta^{\mu\nu}\psi \tag{3.4}$$

Proof of Sufficiency in $d = 2$. Choose $C^{0101}(x) = \psi(x)$ in Eq. (3.2). With the help of Eq. (2.7) and the identity $\varepsilon^{\mu\lambda}\varepsilon^{\rho}_{\mu} = -\eta^{\lambda\rho}$ we find $E^{\mu}_{\mu} = 0$. ■

Proof of Sufficiency in $d \geq 3$. Choose

$$\begin{aligned} C^{\lambda\rho\mu\nu} = {} & \frac{2}{d-2}(\eta^{\lambda\rho}\sigma_{+}^{\mu\nu} - \eta^{\lambda\mu}\sigma_{+}^{\rho\nu} + \eta^{\mu\nu}\sigma_{+}^{\lambda\rho} - \eta^{\rho\nu}\sigma_{+}^{\lambda\mu}) \\ & + \frac{2}{(d-1)(d-2)}(\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\rho}\eta^{\mu\nu})\sigma^{\alpha}_{+\alpha} \end{aligned} \tag{3.5}$$

Long but straightforward calculation using Eq. (2.7) in Eq. (3.1) yields $E^{\mu}_{\mu} = 0$. ■

Proof of Necessary Condition in $d = 2$. If $E^{\mu}_{\mu} = 0$, then, using Eq. (2.7) in (3.2),

$$\partial_\mu[(\Pi^\mu \mathbf{D}\varphi + \Pi_\nu \Sigma^{\mu\nu}\varphi) - \tfrac{1}{2}\partial^\mu C^{0101}] = 0 \tag{3.6}$$

The inverse Poincaré lemma in $d = 2$ demands the existence of a scalar field $\xi(x)$ such that

$$\Pi^\mu \mathbf{D}\varphi + \Pi_\nu \Sigma^{\mu\nu}\varphi = \partial_\nu(\tfrac{1}{2}\eta^{\mu\nu}C^{0101} + \tfrac{1}{2}\varepsilon^{\mu\nu}\xi) \tag{3.7}$$

Therefore $\sigma^{\mu\nu}(x)$ exists where $\sigma_+^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}C^{0101}$. ■

Proof of Necessary Condition for $d \geq 3$. If $E^\mu_\mu = 0$, then from Eqs. (2.7) and (3.1)

$$\partial_\mu[(\Pi^\mu \mathbf{D}\varphi + \Pi_\nu \Sigma^{\mu\nu}\varphi) - \tfrac{1}{2}\partial_\nu C^{\mu\nu\alpha}_\alpha] = 0 \tag{3.8}$$

From the inverse Poincaré lemma, there must exist an antisymmetric tensor $W^{\mu\nu}(x)$ such that

$$\Pi^\mu \mathbf{D}\varphi + \Pi_\nu \Sigma^{\mu\nu}\varphi = \partial_\nu(\tfrac{1}{2}C^{\mu\nu\alpha}_\alpha + W^{\mu\nu}) \tag{3.9}$$

Hence $\sigma^{\mu\nu}(x)$ exists for all $d \geq 3$. ■

The dilatation current takes the neat form $D^\mu = x_\nu E^{\mu\nu}$ and its conservation is an identity now. The improved tensor is also independent of the addition of a total divergence in the Lagrangian.

4. SPECIAL-CONFORMAL INVARIANCE—SUFFICIENT IN $d \geq 3$

The natural extension of the Poincaré group over a d-dimensional flat space-time manifold M is the $[(d + 1)(d + 2)/2]$-parameter conformal group that contains $d(d + 1)/2$ Poincaré generators, one generator for scale transformation, and d generators for special-conformal transformation which leaves the metric tensor invariant up to a local scale factor. In $d \geq 3$, the special-conformal transformation is induced by a coordinate transformation in M

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\alpha x^\alpha + b^2 x^2} \tag{4.1}$$

The commutator of the generators of the special-conformal transformation and translation is a linear combination of the generators of Lorentz and scale transformations (Callan *et al.*, 1970; Coleman and Jackiw, 1971; Francesco *et al.*, 1997). Therefore a Poincaré-invariant field theory in $d \geq 3$ which is also invariant under special-conformal transformation should necessarily be scale invariant.

Theorem 4.1. If a Poincaré-invariant field theory possesses special-conformal symmetry on a flat space-time manifold of $d \geq 3$, then there exists an improved stress-energy tensor.

Proof. Under an infinitesimal special-conformal transformation over flat M, the fields are expected to transform like

$$\varphi(x) \to \varphi'(x') = (\mathbf{I} + \tfrac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta})(\mathbf{I} - \sigma\mathbf{D})\varphi(x) \tag{4.2}$$

where $\omega_{\alpha\beta} = 2(x_\alpha b_\beta - x_\beta b_\alpha)$ and $\sigma = 2b_\mu x^\mu$. With the help of the equation of motion, scale invariance implies

$$\frac{\partial \mathscr{L}}{\partial \varphi}\mathbf{D}\varphi + \Pi^\mu \mathbf{D}\partial_\mu\varphi + \Pi^\mu\partial_\mu\varphi = d\mathscr{L} \tag{4.3}$$

According to Noether's theorem, the weakest requirement for the existence of a conserved current due to an infinitesimal diffeormorphism $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$, $|\varepsilon^\mu(x)| << |x^\mu|$ is the existence of a vector field $V^\mu(x)$ such that $\delta\mathscr{L} + \partial_\mu(\varepsilon^\mu\mathscr{L}) = \partial_\mu V^\mu$ (Callan *et al.*, 1970; Coleman and Jackiw, 1971). For special-conformal symmetry

$$\partial\mathscr{L} + \partial_\mu(\varepsilon^\mu\mathscr{L}) = -2b_\mu(\Pi^\mu\mathbf{D}\varphi + \Pi_\nu\Sigma^{\mu\nu}\varphi) \tag{4.4}$$

b^μ is an arbitrary constant vector. Therefore there must exist a tensor field $\sigma^{\mu\nu}(x)$ such that $\Pi^\mu\mathbf{D}\varphi + \Pi_\nu\Sigma^{\mu\nu}\varphi = \partial_\nu\sigma^{\mu\nu}$. Hence the improved tensor exists. ■

The special-conformal current takes a neat form and it is conserved identically when expressed in terms of the improved tensor

$$C^{\mu\nu} = E^\mu_\alpha\,(2x^\alpha x^\nu - \eta^{\alpha\nu}x^2) \tag{4.5}$$

The theorem does not hold true for $d = 2$. In 2 dimensions there exist an infinite variety of locally conformal transformations (not well defined everywhere in M) that map various submanifolds of M into itself. Therefore the special-conformal transformation is not uniquely defined in $d = 2$. If it is possible to generalize the flat space-time action of a conformally invariant field theory to the corresponding Riemannian manifold via minimal coupling ($\eta_{\mu\nu} \to g_{\mu\nu}(x)$, $\partial_\mu \to \nabla_\mu$, $\nabla_\mu g_{\alpha\beta} = 0$, $d^dx \to d^dx\sqrt{|g|}$) such that the action on the Riemannian manifold becomes general-conformal invariant, then $E^{\mu\nu} \equiv T^{\mu\nu}$ ($\sigma^{\mu\nu} \equiv 0$) for all $d \geq 2$.

5. EXAMPLES

Real Scalar Field. The classical scalar field theory described by

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - k\varphi^{2d/(d-2)} \tag{5.1}$$

(physically dimensionless $k \in \mathfrak{R}$, $k = 0$ for $d = 2$) defines a conformally invariant action over flat M (Wald, 1984). Due to Theorem 4.1 an improved tensor exists [$\sigma^{\mu\nu} = \frac{1}{2}(d/2 - 1)\eta^{\mu\nu}\varphi^2$] and we have

$$E^{\mu\nu} = T^{\mu\nu} + \frac{1}{4}\left(\frac{d-2}{d-1}\right)(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\varphi^2 \tag{5.2}$$

Vector Field. The scale-invariant Lagrangian

$$\mathscr{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}, \qquad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{5.3}$$

describes a conformally invariant action only in $d = 4$ (general-conformal invariant implies $\sigma^{\mu\nu} \equiv 0$, $E^{\mu\nu} \equiv T^{\mu\nu}$), but a nontrivial improved stress-energy tensor exists for all $d \geq 3$ provided the gauge symmetry has been broken in favor of the Lorentz gauge $\partial_\mu A^\mu = 0$ in $d \neq 4$. Only then $\sigma^{\mu\nu} = [(4 - d)/8\pi](\frac{1}{2}\eta^{\mu\nu} A^\alpha A_\alpha - A^\mu A^\nu)$ and

$$\begin{aligned} E^{\mu\nu} = T^{\mu\nu} &+ \frac{d(4-d)}{16\pi(d-1)(d-2)}(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A^\alpha A_\alpha \\ &+ \frac{d-4}{8\pi(d-2)}[\Box(A^\mu A^\nu) + \eta^{\mu\nu}\partial_\alpha\partial_\beta(A^\alpha A^\beta) \\ &- \partial^\mu\partial_\alpha(A^\alpha A^\nu) - \partial^\nu\partial_\alpha(A^\alpha A^\mu)] \end{aligned} \tag{5.4}$$

This example explicitly shows that conformal invariance over a flat space-time manifold is not necessary for the existence of an improved stress-energy tensor. With the help of equations of motion, $E^{\mu\nu} = 0$ in $d = 2$.

Dirac Bispinor Field. The Lagrangian

$$\mathscr{L} = \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi \tag{5.5}$$

describes a general-conformal invariant action for all space-time dimensions. Here

$$E^{\mu\nu} \equiv T^{\mu\nu} = \frac{i}{4}\bar{\psi}(\gamma^\mu \overleftrightarrow{\partial}^\nu + \gamma^\nu \overleftrightarrow{\partial}^\mu)\psi \tag{5.6}$$

for all $d \geq 2$ ($\sigma^{\mu\nu} \equiv 0$).

6. CONCLUSION

We proved that the conserved symmetric stress-energy tensor of a Poincaré-invariant classical field theory over a flat space-time manifold can be improved to a conserved traceless symmetric tensor for a scale-invariant action if and only if the trace of the conventional symmetric tensor is a second-order divergence in $d \geq 3$ and a Laplacian in $d = 2$. Although there is compelling evidence for its existence from quantum theories, before this paper there was no general proof or known method of construction of the improved tensor in $d = 2$ (Francesco *et al.*, 1997). The classical scale, or in general, conformal invariance of a massless field theory with physically dimensionless coupling constants is broken by the quantum corrections coming from the renormalization of the coupling constants which leads to trace anomaly in the improved tensor. However, in $d = 2$, there exist special systems (e.g., 2-dimensional nonlinear sigma model) where the anomaly at the quantum level disappears and the quantum theory becomes invariant to conformal mapping. Therefore although in general it is not possible to promote the classical symmetries at the quantum level for arbitrary space-time dimensions, in the special case of $d = 2$ the existence of an improved tensor in scale-invariant classical Poincaré field theory has significance in quantum field theory.

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